COMPLEX ANALYSIS HOMEWORK 6

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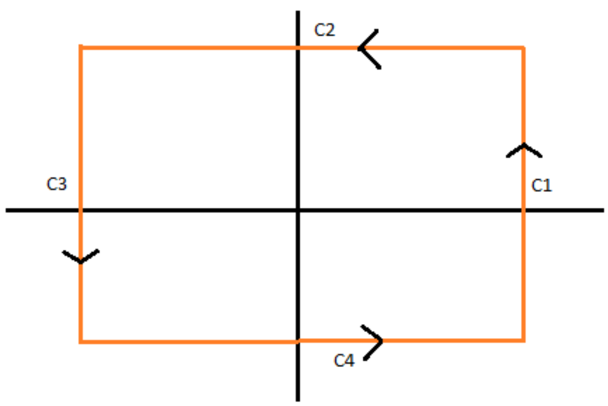
1. Problem 1

Define $f(z) := \frac{\pi \cot(\pi z)}{z^2+a^2}$. Consider

$$\int_C f(z)dz$$

Over a rectangular contour C, as shown below.



With C_1 a vertical line with $\text{Re}(z) = n + 1/2$ and similarly C_3 a vertical line at $\text{Re}(z) = -(n+1/2)$, $n \in \mathbb{Z}$ is any integer larger than a. C_2 is a line such that $\text{Im}(z) = n$ and similarly C_4 is such that $\text{Im}(z) = -n$.

By Cauchy's Residue theorem, note that the cotangent function has poles at the points πk, $k \in \mathbb{Z}$, and f has two additional simple poles at $\pm ia$. Using this, we have:

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$$\int_C f(z)dz = 2\pi i\Big(\sum_{k\in\mathbb{Z}} \text{Res}(f,k) + \text{Res}(f,ia) + \text{Res}(f,-ia)\Big)$$

By definition,

$$\begin{aligned}\text{Res}(f,k) &= \lim_{z\to k}(z-k)\frac{\pi\cot(\pi z)}{z^2+a^2} \\ &= \frac{\lim_{z\to k}\pi\cos(\pi z)}{(\sin(\pi z))'|_k(z^2+a^2)} \\ &= \frac{\cos(\pi k)}{\cos(\pi k)(z^2+k^2)} \\ &= \frac{1}{z^2+k^2}\end{aligned} \tag{1.1}$$

Also,

$$\begin{aligned}\text{Res}(f,ia) &= \lim_{z\to ia}(z-ia)\frac{\pi\cot(\pi z)}{z^2+a^2} \\ &= \lim_{z\to ia}\frac{\pi\cot(\pi z)}{(z+ia)} \\ &= \frac{\pi\cot(\pi ai)}{2ia} \\ &= -\frac{\pi\coth(\pi a)}{2a}\end{aligned} \tag{1.2}$$

Similarly by the above we see that $\text{Res}(f,-ia) = -\frac{\pi\coth(\pi a)}{2a}$ as well. Thus, we have:

$$\begin{aligned}\frac{1}{2\pi i}\int_C f(z)dz &= \sum_{k=-n}^{n}\frac{1}{k^2+a^2} - \frac{\pi\coth(\pi a)}{a} \\ &= 2\sum_{k=0}^{n}\frac{1}{k^2+a^2} - \frac{1}{a^2} - \frac{\pi\coth(\pi a)}{a}\end{aligned} \tag{1.3}$$

We want to bound $\int_C f(z)dz$ and let $n\to\infty$. On C_1 we have that $z = (n+1/2)+ti$, where $t\in(-n,n)$. Thus,

$$\begin{aligned}\Big|\int_{C_1} f(z)dz\Big| &\leq \int_{-n}^{n} \frac{|\cot(\pi(n+1/2+it))|}{|((n+1/2)+it)^2+a^2|}|dz| \\ &\leq \int_{-n}^{n} \frac{|e^{-t}-e^{t}|}{|e^{-t}+e^{t}|}\frac{1}{|n^2+a^2|}|dz| \\ &\leq 2n\frac{1}{n^2+a^2} \to 0\end{aligned} \tag{1.4}$$

As $n \to \infty$. Note that this also takes care of the case for the contour C_3, since we merely set $(n+1/2) \mapsto -(n+1/2)$, and the rest is identical to above. Now, on C_2:

$$\begin{aligned}\Big|\int_{C_2} f(z)dz\Big| &\leq \int_{-(n+1/2)}^{n+1/2} \frac{|\cot(\pi(t+in))|}{|(t+in)^2+a^2)|}|dz| \\ &\leq \int_{-(n+1/2)}^{n+1/2} \frac{|e^{-n}|+|e^{n}|}{||e^{-n}|-|e^{n}||}\frac{1}{n^2+a^2}|dz| \\ &\leq (2n+1)\frac{e^{-n}+e^{n}}{|e^{-n}-e^{n}|}\frac{1}{n^2+a^2} \to 0\end{aligned} \tag{1.5}$$

As $n \to \infty$, where we note that $\frac{e^{-n}+e^{n}}{|e^{-n}-e^{n}|}$ is bounded for n positive. Again, the above work handles the case for C_4 since we merely change $in \mapsto -in$. Using the above, we can let $n \to \infty$ in (1.3) to see that $\frac{1}{2\pi i}\int_C f(z)dz \to 0$, so that:

$$\begin{aligned}0 = &= 2\sum_{k=0}^{\infty} \frac{1}{k^2+a^2} - \frac{1}{a^2} - \frac{\pi\coth(\pi a)}{a} \\ \implies &\sum_{k=0}^{\infty} \frac{1}{k^2+a^2} = \frac{1}{2a^2} + \frac{\pi\coth(\pi a)}{2a}\end{aligned} \tag{1.6}$$

As desired.

2. Problem 2

Using the result of the previous problem, set $a = 1$. We then see:

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1} = \frac{1}{2} + \frac{\pi \coth(\pi)}{2}$$

3. Problem 3

We use the exact same contour C as presented in Problem 1. This time, define $f(z) := \frac{\pi \cot(\pi z)}{(2z-1)^2}$. Then,

$$\frac{1}{2\pi i} \int_C f(z)dz = \sum_{k=-n}^{n} \text{Res}(f,k) + \text{Res}(f,1/2)$$

We then find:

$$\begin{aligned} \text{Res}(f,k) &= \lim_{z\to k} (z-k) \frac{\pi \cot(\pi z)}{(2z-1)^2} \\ &= \frac{\lim_{z\to k} \pi \cos(\pi z)}{(\sin(\pi z))'|_k (2z-1)^2} \\ &= \frac{\cos(\pi k)}{\cos(\pi k)(2z-1)^2} \\ &= \frac{1}{(2k-1)^2} \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} \text{Res}(f,1/2) &= \lim_{z\to 1/2} \frac{d}{dz}(z-1/2)^2 \frac{\pi \cot(\pi z)}{4(z-1/2)^2} \\ &= \lim_{z\to 1/2} -\frac{\pi^2 \csc^2(\pi z)}{4} \\ &= -\frac{\pi^2}{4} \end{aligned} \tag{3.2}$$

Using this,

$$\begin{aligned} \frac{1}{2\pi i}\int_C f(z)dz &= \sum_{k=-n}^{n}\frac{1}{(2k-1)^2} - \frac{\pi^2}{4} \\ &= 2\sum_{k=0}^{n}\frac{1}{(2k-1)^2} - 2 - \frac{\pi^2}{4} \end{aligned} \tag{3.3}$$

Similar to in Problem 1, we now want to show $\int_C f(z)dz \to 0$ as $n \to \infty$. On C_1,

$$\begin{aligned} \Big|\int_{C_1} f(z)dz\Big| &\leq \int_{-n}^{n} \frac{|\cot(\pi(n+1/2+it))|}{|((2n+1+2it-1)|^2}|dz| \\ &\leq \int_{-n}^{n} \frac{|e^{-t}-e^{t}|}{|e^{-t}+e^{t}|}\frac{1}{n^2+t^2}|dz| \\ &\leq 2n\frac{1}{n^2} \to 0 \end{aligned} \tag{3.4}$$

As $n \to \infty$. Again, this shows that the same happens on C_3. On C_2:

$$\begin{aligned} \Big|\int_{C_2} f(z)dz\Big| &\leq \int_{-(n+1/2)}^{n+1/2} \frac{|\cot(\pi(t+in))|}{|(2t+2in-1)^2|}|dz| \\ &\leq \int_{-(n+1/2)}^{n+1/2} \frac{|e^{-n}|+|e^{n}|}{||e^{-n}|-|e^{n}||}\frac{1}{t^2+n^2}|dz| \\ &\leq (2n+1)\frac{e^{-n}+e^{n}}{|e^{-n}-e^{n}|}\frac{1}{n^2} \to 0 \end{aligned} \tag{3.5}$$

Where we have again used the fact that $\frac{e^{-n}+e^{n}}{|e^{-n}-e^{n}|}$ is bounded to conclude that the entire expression tends to 0 as $n \to \infty$. This also takes care of the case for the contour C_4.

Combining the above with (3.3), we use that $\int_C f(z) \to 0$, so that:

$$\begin{aligned} 0 &= 2\sum_{k=0}^{\infty}\frac{1}{(2k-1)^2} - 2 - \frac{\pi^2}{4} \\ \implies &\sum_{k=0}^{\infty}\frac{1}{(2k-1)^2} = 1 + \frac{\pi^2}{8} \end{aligned} \tag{3.6}$$

And we are done.